United Kingdom
Mathematics Trust

# Intermediate Mathematical Olympiad

## Cayley paper

**Thursday 17 March 2022**





*England & Wales: Year 9 or below*
*Scotland: S2 or below*
*Northern Ireland: Year 10 or below*

These problems are meant to be challenging! The earlier questions tend to be easier; later questions tend to be more demanding.

Do not hurry, but spend time working carefully on one question before attempting another.

Try to finish whole questions even if you cannot do many: you will have done well if you hand in full solutions to two or more questions.

You may wish to work in rough first, then set out your final solution with clear explanations and proofs.

## Instructions

1. Do not open the paper until the invigilator tells you to do so.
2. Time allowed: **2 hours**.
3. The use of blank or lined paper for rough working, rulers and compasses is allowed; **squared paper, calculators and protractors are forbidden**.
4. You should write your solutions neatly on A4 paper. Staple your sheets together in the top left corner with the Cover Sheet on top and the questions in order.
5. Start each question on a fresh A4 sheet. **Do not hand in rough work**.
6. Your answers should be fully simplified, and exact. They may contain symbols such as $\pi$, fractions, or square roots, if appropriate, but not decimal approximations.
7. You should give full written solutions, including mathematical reasons as to why your method is correct. Just stating an answer, even a correct one, will earn you very few marks; also, incomplete or poorly presented solutions will not receive full marks.

Enquiries about the Intermediate Mathematical Olympiad should be sent to:

*UK Mathematics Trust, School of Mathematics, University of Leeds, Leeds LS2 9JT*

☎ 0113 365 1121    challenges@ukmt.org.uk    www.ukmt.org.uk

- ◇ *Do not hurry, but spend time working carefully on one question before attempting another.*
- ◇ *Try to finish whole questions even if you cannot do many.*
- ◇ *You will have done well if you hand in full solutions to two or more questions.*
- ◇ *Your answers should be fully simplified, and exact. They may contain symbols such as* π*, fractions, or square roots, if appropriate, but* not *decimal approximations.*
- ◇ *Give full written solutions, including mathematical reasons as to why your method is correct.*
- ◇ *Just stating an answer, even a correct one, will earn you very few marks.*
- ◇ *Incomplete or poorly presented solutions will not receive full marks.*
- ◇ *Do* not *hand in rough work.*

**1.** The numbers 62, 63, 64, 65, 66, 67, 68, 69 and 70 are divided by, in some order, the numbers 1, 2, 3, 4, 5, 6, 7, 8 and 9, resulting in nine integers. The sum of these nine integers is $S$. What are the possible values of $S$?

**2.** A palindromic number is a positive integer which reads the same when its digits are reversed, for example 269 962. Find all six-digit palindromic numbers that are divisible by 45.

**3.** Consider the equation $0.abcd + 0.efgh = 1$, where each letter stands for a digit from 1 to 8 inclusive.

(a) Suppose each letter stands for a different digit. Prove that there are no solutions.

(b) Suppose instead that digits may be repeated. How many solutions are there? (You may give your final answer as a product of prime numbers if you wish.)

*Note that* $(0.abcd, 0.efgh)$ *and* $(0.efgh, 0.abcd)$ *are considered to be the same solution.*

**4.** The regular octagon $ABCDEFGH$ is inscribed in a circle. Points $P$ and $Q$ are on the circle, with $P$ between $C$ and $D$, such that $APQ$ is an equilateral triangle. It is possible to inscribe a regular $n$-sided polygon, one of whose sides is $PD$, in the circle. What is the value of $n$?

**5.** Consider equations of the form $ax + b = c$, where $a, b$ and $c$ are integers such that one is the sum of the other two and $a$ is non-zero. What are the possible integer values of $x$?

**6.** Seth has nine stones: three painted blue, three painted red and three painted yellow. The blue stones are labelled 1, 2 and 3, as are the red stones and the yellow stones. He builds a vertical tower with three stones, putting one on top of another.

Three stones form a *set* if any of the following hold:

(i) They all have the same colour;

(ii) They are all labelled with the same number;

(iii) They all have different colours;

(iv) They are all labelled with different numbers.

In how many ways can he build a tower that **avoids** creating a set?

 